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ELASTIC STRESS SINGULARITY AT CONICAL INCLUSIONS

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Abstract—The nature of singularities at the vertex of a conical inclusion bonded into a conical notch is found for the problem of axially symmetric stress distribution. The solution is obtained from an eigenvalue formulation based upon expansions in terms of spherical harmonics. For certain ranges of elastic constants the singularities are found to be complex valued.

INTRODUCTION

The strength of singularities at the vertex of conical notches and rigid inclusions in the axisymmetric state of stress has been recently analyzed by Bažant and Keer[1]. The purpose of this note is to extend their analysis based on spherical harmonics to the case in which the inclusion and the matrix are both elastic. The problems involving notches and rigid inclusions remain limiting cases of the bimaterial one under consideration. However, the stresses in the neighborhood of the vertex exhibit oscillatory behavior for certain combinations of material constants in the bimaterial problem.

METHOD OF SOLUTION

Let the region occupied by the material with rigidity modulus G_1 and Poisson's ratio ν_1 be referred as Region 1. In spherical polar coordinates (R, ϕ_o, ϕ), Region 1 is defined by $0 < R < \infty$, $0 \le \phi < \beta$, $0 \le \phi_o < 2\pi$. The complement of Region 1 in the whole space is Region 2 occupied by the material with rigidity modulus G_2 and Poisson's ratio ν_2. Introduce spherical polar coordinates (R, θ_o, θ) in Region 2 in such a way that the common boundary of the two regions is the conical surface $\theta = \pi - \beta$, $0 \le \theta_0 < 2\pi$ (see Fig. 1).

Following Thompson and Little[2], let the displacement vector for Region 1 be written

$$u_R^{(1)}(R, \phi) = R^\lambda [C_1(1 - k_1 - k_1\lambda)\mu P_\lambda(\mu) + (1 + \lambda)D_1 P_{\lambda+1}(\mu)], \tag{1}$$

$$u_\phi^{(1)}(R, \phi) = -R^\lambda [(D_1 - k_1 C_1)P'_{\lambda+1}(\mu) + C_1(1 + k_1\lambda)P_\lambda(\mu)] \sin \phi \tag{2}$$

where prime denotes derivative with respect to the argument of the Legendre function and we have

$$k_1 = 1/(4 - 4\nu_1), \qquad \mu = \cos \phi. \tag{3}$$

The unspecified quantities C_1, D_1 are independent of the coordinate variables (R, ϕ_o, ϕ) but, in general, they are functions of Poisson ratios and the parameter λ. Similarly, the displacement vector for Region 2 may be written

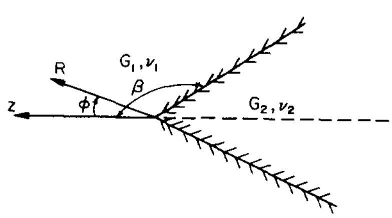

Fig. 1. Geometry and coordinate system.

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$$u_R^{(2)}(R,\theta) = R^\lambda[C_2(1-k_2-k_2\lambda)\delta P_\lambda(\delta)+(1+\lambda)D_2P_{\lambda+1}(\delta)], \tag{4}$$

$$u_\theta^{(2)}(R,\theta) = -R^\lambda[(D_2-k_2C_2)P'_{\lambda+1}(\delta)+C_2(1+k_2\lambda)P_\lambda(\delta)]\sin\theta, \tag{5}$$

and

$$k_2 = 1/(4-4\nu_2), \qquad \delta = \cos\theta \tag{6}$$

Except for the continuity conditions on the common boundary of the two regions, no other conditions on the radial rays will be specified, and so an infinite number of displacement vectors of the form (1)–(6) are naturally expected to be allowable. However, in a sufficiently small neighborhood of the apex of the conical boundary, the displacement vector that corresponds to the smallest real part of the parameter λ prevails. Consequently, determination of the smallest allowable $Re(\lambda)$ is of particular interest, especially when $Re(\lambda)<1$ because the stress components are then unbounded near the apex $R=0$.

With an obvious notation, the continuity conditions on the common conical boundary of Regions 1 and 2 are given by

$$u_R^{(1)}(R,\beta) = u_R^{(2)}(R,\pi-\beta), \tag{7}$$

$$u_\phi^{(1)}(R,\beta) = -u_\theta^{(2)}(R,\pi-\beta), \tag{8}$$

$$\sigma_{\phi\phi}^{(1)}(R,\beta) = \sigma_{\theta\theta}^{(2)}(R,\pi-\beta), \tag{9}$$

$$\sigma_{R\phi}^{(1)}(R,\beta) = -\sigma_{R\theta}^{(2)}(R,\pi-\beta). \tag{10}$$

The stress components corresponding to the displacement vectors (1)–(6) can be easily derived by means of the stress-displacement relations. Then the continuity conditions (7)–(10) yield a set of four linear homogeneous equations in terms of the four unknowns C_1, D_1, C_2 and D_2. For a non-trivial solution of these equations the determinant of the coefficient matrix, say, $[A_{ij}]$ must vanish. The elements of the matrix may be written

$$A_{11}(\beta,\nu_1,\lambda) = \{1-k_1(\lambda+1)\}\mu_o P_\lambda(\mu_o), \tag{11}$$

$$A_{21}(\beta,\nu_1,\lambda) = (1+k_1\lambda)(1-\mu_o^2)P_\lambda(\mu_o)-k_1(\lambda+1)\{P_\lambda(\mu_o)-\mu_oP_{\lambda+1}(\mu_o)\}, \tag{12}$$

$$\begin{aligned}A_{31}(\beta,\nu_1,\lambda) = k_1[&\{(\lambda+1)^2+(\lambda+1)+(1-2\nu_1)\}\mu_o(1-\mu_o^2)P_\lambda(\mu_o)\\ &-2(1-\nu_1)(\lambda+1)(1-\mu_o^2)P_{\lambda+1}(\mu_o)\\ &-\mu_o(\lambda+1)\{P_\lambda(\mu_o)-\mu_oP_{\lambda+1}(\mu_o)\}],\end{aligned} \tag{13}$$

$$\begin{aligned}A_{41}(\beta,\nu_1,\lambda) = k_1[\{\lambda^2-2(1-\nu_1)\}(1-\mu_o^2)P_\lambda(\mu_o)+(\lambda+1)\{-\lambda+2(1-\nu_1)\}\{P_\lambda(\mu_o)\\ -\mu_oP_{\lambda+1}(\mu_o)\}],\end{aligned} \tag{14}$$

$$A_{12}(\beta,\nu_1,\lambda) = (1+\lambda)P_{\lambda+1}(\mu_o), \tag{15}$$

$$A_{22}(\beta,\nu_1,\lambda) = (1+\lambda)\{P_\lambda(\mu_o)-\mu_oP_{\lambda+1}(\mu_o)\}, \tag{16}$$

$$A_{32}(\beta,\nu_1,\lambda) = -(\lambda+1)^2(1-\mu_o^2)P_{\lambda+1}(\mu_o)+\mu_o(\lambda+1)\{P_\lambda(\mu_o)-\mu_oP_{\lambda+1}(\mu_o)\}, \tag{17}$$

$$A_{42}(\beta,\nu_1,\lambda) = \lambda(\lambda+1)\{P_\lambda(\mu_o)-\mu_oP_{\lambda+1}(\mu_o)\}, \tag{18}$$

and

$$A_{1j}(\beta,\nu_2,\lambda) = A_{1(j-2)}(\pi-\beta,\nu_2,\lambda), \qquad j=3,4 \tag{19}$$

$$A_{2j}(\beta,\nu_2,\lambda) = -A_{2(j-2)}(\pi-\beta,\nu_2,\lambda), \qquad j=3,4 \tag{20}$$

$$A_{3j}(\beta,\nu_2,\lambda) = \frac{G_2}{G_1}A_{3(j-2)}(\pi-\beta,\nu_2,\lambda), \qquad j=3,4 \tag{21}$$

$$A_{4j}(\beta,\nu_2,\lambda) = -\frac{G_2}{G_1}A_{4(j-2)}(\pi-\beta,\nu_2,\lambda), \qquad j=3,4 \tag{22}$$

where

$$\mu_o = \cos\beta, \qquad k_1 = 1/(4-4\nu_1), \qquad k_2 = 1/(4-4\nu_2). \tag{23}$$

The requirement of zero determinant for the matrix $[A_{ij}]$, that is,

$$\mathrm{Det}\,[A_{ij}] = 0, \tag{24}\dagger$$

†The authors wish to mention that eqn (16) in Ref. [1], which corresponds to eqn (24) here, contains a misprint which may be corrected by replacing $2(1-\nu)\lambda^2$ in the last line by $2(1-2\nu)x^2$. Also, the power λ of r in eqn (21) should be replaced by $-\lambda-1$.

Table 1. Singularity Values λ for Range of Elastic Constants and Angle β (All values of λ are real)

G_1/G_2	ν_1	ν_2	0.51π	$7\pi/12$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$11\pi/12$	0.97π
0.1	0.40	0.0	.9973	.9748	.9066	.7508	.6281	.6608	.9133
0.1	0.35	0.0	.9939	.9458	.8568	.7206	.6131	.6540	.9123
0.1	0.30	0.0	.9914	.9261	.8267	.6991	.6012	.6483	.9115
0.1	0.25	0.0	.9894	.9114	.8053	.6826	.5914	.6435	.9108
0.1	0.0	0.50	.9810	.8521	.7179	.5966	.5092	.5450	.8583
0.1	0.25	0.25	.9874	.8962	.7812	.6560	.5604	.5990	.8875
0.25	0.25	0.25	.9908	.9234	.8363	.7455	.7007	.8019	.9650
4.00	0.25	0.25	.9785	.8722	.8514	.8962	.9524	.9886	.9986
4.00	0.50	0.0	.9685	.8061	.7459	.7857	.8839	.9711	.9964
4.00	0.0	0.25	.9868	.9225	.9319	.9618	.9841	.9962	.9995
4.00	0.0	0.30	.9883	.9403	.9565	.9806	.9925	.9982	.9998

Table 2. Singularity Values λ for Range of Elastic Constants and Angle β (λ complex for certain range of angle β)

G_1/G_2	ν_1	ν_2	0.51π	$7\pi/12$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$11\pi/12$	0.97π
0.1	0.50	0.0	1.0128 .0057i	1.0857 .0894i	.9943 .1556i	.8908 .0747i	.6741 .0	.6792 .0	.9157 .0
4.0	0.0	0.40	.9939 .0061i	.9690 .0544i	1.0002 .1039i	1.0965 .0623i	1.0239 .0	1.0052 .0	1.0007 .0
4.0	0.0	0.50	.9914 .0131i	.9458 .0995i	.9533 .1861i	1.0532 .2284i	1.1208 .0650i	1.0424 .0	1.0064 .0

is an eigenvalue problem for λ when the Poisson ratios ν_1, ν_2, the ratio of the shear moduli G_1/G_2 and the cone angle 2β are prescribed. In the present context, the roots λ with $Re(\lambda) < 1$ are the only ones of interest.

NUMERICAL RESULTS

Equation (24) was solved for the eigenvalue, λ, for several values of the elastic constants. The numerical results are presented in Tables 1 and 2. The angles range from $(\pi/2)$ to π, where the first value represents two bonded half spaces (no singularity). The subscript "1" designates material having a conical cavity of semi vertex angle, π-β, which is bonded to material of subscript "2" in the form of a cone of semi vertex angle π-β. Table 1 shows values of the ratio of shear moduli and combinations of Poisson's ratio such that the singularity values are real. In Table 2 some values are given in which the singularity exponent will be complex. Cases in which the singularity value is complex are somewhat limited in range and are seen to depend upon extreme differences in the elastic constants for matrix and inclusion and also upon the cone angle.

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